Preface

When a researcher is faced with experimental results seeming to obey a deterministic law, usually a specific mathematical model is built, tested, and validated. Many models are indeed formulated as recurrent relations defining iterated maps. These are models describing dynamics of populations with non-overlapping generations, as well as biological, economical, and industrial systems subject to periodic environmental influence. With varying parameters in a model, different behavior can be observed, providing explanations of the experimental results. To actually analyze such a model one needs to draw on theoretical knowledge but also appropriate numerical methods. Preferably these will be available through user-friendly software. With this goal we have worked on developing theory and algorithms for our MATLAB® toolbox MATCONTM over the past decade.

This book covers discrete-time dynamical systems generated by iterated nonlinear maps. In particular, it explains how their dynamics change under variation of parameters, which is a subject of bifurcation theory. We present these topics via a systematic treatment of bifurcations of fixed points and cycles up to and including cases in which two system parameters are involved. Theoretical results for two-parameter bifurcations have been obtained during the past 40 years. There are a number of recent developments available to experts in the field through research papers only. This textbook fills this gap by presenting the theory systematically and consistently, from an introductory level up to current research topics.

Through our recent work, the work of collaborators, and other researchers in the field, we have obtained a fairly complete understanding of local bifurcations of maps and can apply these results to concrete models. Local bifurcation theory gives good indicators and descriptions of how a certain model behaves, but in practice global characteristics are used too. Therefore, our treatment also includes several of these complementary methods, such as Lyapunov exponents, invariant manifolds and homoclinic structures, and parts of chaos theory.

The power of the developed theory, methods, and computer algorithms will be illustrated on both elementary and more realistic models. We provide step-by-step tutorials to introduce the reader to MatcontM. Here, we focus on the functionality using rather simple dynamical models defined by one- and two-dimensional maps. These tutorials illustrate how the general numerical methods described in the book and implemented in MatcontM can be used. Even in the simplest situations, this provides useful insight. In addition, we show how to study more complicated models from engineering, ecology, and economics. We provide code to reproduce the numerical results using our free toolbox, MatcontM.

This book is written for those who study discrete-time dynamical models that frequently appear in various scientific disciplines. It is accessible not only for applied mathematicians, but also for researchers with a moderate mathematical background (e.g., basic differential equations and numerical analysis). Researchers from different areas can use it as a reference text for some advanced topics. Some results will be new even to experts. Active support for the software has given us valuable feedback about where users experience difficulties. Moreover, our teaching experience has shown that parts of this book can be used in regular and advanced (post-)graduate courses on nonlinear dynamics and mathematical modeling. This book can be used as

- material for systematic study of bifurcation theory of maps, if you read it from beginning to end;
- a theoretical reference book for specific topics, e.g., a particular bifurcation;
- description of numerical bifurcation methods for maps that one could implement her/himself;
- a user-guide for particular software amenable to all theory and methods, including step-by-step tutorials; and
- a source of case studies ranging from elementary to recent research topics.

The famous notion of a Poincaré map intimately relates our exposition also to continuous-time dynamics of ordinary differential equations (ODEs), and thus the material is also useful for applications with limit cycles.

There are a number of books treating local codim 1 and 2 bifurcation of maps theoretically. Such books are either entirely devoted to maps (Neimark (1972); Iooss (1979); Mira (1987); Devaney (2003)), or have many chapters about maps (Guckenheimer and Holmes (1990); Arnold (1983); Arrowsmith and Place (1990); Kuznetsov (2004); Wiggins (2003)). It should be noted that generic two-parameter bifurcations of fixed points and cycles also involve global bifurcations leading to fractal parameter portraits and chaotic dynamics. A detailed treatment of these complications is usually only discussed at a theoretical level. The number of books on numerical analysis of maps is very limited, (e.g., Nusse and Yorke (1998); Abraham, Gardini, and Mira (1997)), but these focus on the visualization of the phase space of planar maps and noninvertibility. To the best of our knowledge, no existing book systematically describes numerical techniques for continuation, normal forms, invariant manifolds, and Lyapunov exponents to study maps depending on several parameters. We aim to fill this gap. This book provides theoretical and practical details to study the dynamics in generic two-parameter families of maps. In particular, we not only describe dynamics of approximating ODEs, but systematically study effects of their nonsymmetric perturbations, including quasi-periodic bifurcations. This will be helpful to elucidate the route to chaos in many models. The book will also teach the reader how to use the MATLAB software toolbox MatcontM that implements the developed numerical algorithms.

In Part One we first introduce analytical techniques that will be used later to study bifurcations. We briefly summarize without proof well-known results on local bifurcations in the one-parameter families following Kuznetsov (2004). The parameter-dependent normal forms on the center manifolds are given. We treat only those global bifurcations that appear near codim 2 bifurcations studied later, i.e., homoclinic tangencies, and some quasi-periodic bifurcations of closed invariant curves and 2D tori. Then we systematically present with proofs results on normal form analysis for all 11 local bifurcations of codim 2. Our exposition is complete, yet brief for the simplest cases, which are also treated by Arrowsmith and Place (1990) and Kuznetsov (2004), i.e., cusp, generalized period-doubling bifurcations, fold-flip, and strong resonances. We provide complete proofs and correct mistakes occurring in the literature. We also treat the most complicated codim 2 cases (flip-NS, fold-NS, and double NS bifurcations), which currently have been studied only in journal articles. In all cases, we derive critical and parameter-dependent normal forms and study their local bifurcations, then we obtain relevant approximating ODEs and analyze their local and global bifurcations, thus gaining insight into the main features of canonical local bifurcation diagrams. Moreover, we include new results on homoclinic and quasi-periodic bifurcations near codim 2 points by considering representative nonsymmetric perturbations of the truncated normal forms. Finally, we derive explicit formulas for the normal form coefficients of the restricted maps to the relevant center manifolds, which are then used to construct efficient predictors for codim 1 local bifurcation curves from codim 2 points.

Part Two is devoted to various algorithms for numerical bifurcation analysis of smooth maps, combining continuation techniques with normal form computations and constructing of Lyapunov charts. While modern methods for numerical bifurcation analysis of ODEs are systematically presented in several texts (e.g., Kuznetsov (2004); Govaerts (2000)), no single book is available with such methods for maps. These algorithms are scattered across journal publications, including ours, and we collect them here using uniform notation. We also discuss methods to compute the necessary partial derivatives, including automatic differentiation, and also for maps obtained via numerical integration. Then we describe the functionality of MatcontM and provide detailed step-by-step tutorials on how to use this toolbox. Here, we use simple models, e.g., the Ricker map and the delayed logistic map.

In Part Three we demonstrate the effectiveness of the developed methods and software MatcontM on more complicated models that range from the generalized Hénon map (which plays an important role in theoretical analysis of codim 2 homoclinic bifurcations of maps) to models from engineering (adaptive control map) and economics (duopoly model of Kopel). Practically all results – some of which are novel – are obtained using MatcontM. The last example (SEIR epidemic model) is a periodically forced ODE system. In this case, we apply the developed techniques and software to the numerically computed Poincaré return map.

While working on the topics included in this book, we collaborated with many colleagues and friends. First of all, we want to thank Willy Govaerts (Ghent University, Belgium) for long-term collaboration on developing numerical methods and interactive software for the analysis of continuous- and discrete-time dynamical systems, and for developing the MATLAB bifurcation toolboxes Matcont and MatcontM. We thank Stephan van Gils (University of Twente, Enschede, the Netherlands) for supporting this project. We acknowledge Odo Diekmann and Ferdinand Verhulst (Utrecht University, the Netherlands) for discussions on numerous topics. We are also thankful to Eusebius Doedel, Bernd Krauskopf, Hinke Osinga, and Renato Vitolo for stimulating discussions of various aspects of numerical bifurcation analysis of maps and ODEs. We acknowledge contributions of the (post-)graduate students we supervised, namely Reza Khoshsiar Ghaziani, Niels Neirynck, and Matthias Aengenheyster.