Chapter 1

Introduction to Differential Equations

1.1 Definitions and Terminology

3. Fourth order; linear

6. Second order; nonlinear because of R^2

9. First order; nonlinear because of $\sin\left(\dfrac{dy}{dx}\right)$

12. Writing the differential equation in the form $u(dv/du) + (1+u)v = ue^u$, we see that it is linear in v. However, writing it in the form $(v + uv - ue^u)(du/dv) + u = 0$, we see that it is nonlinear in u.

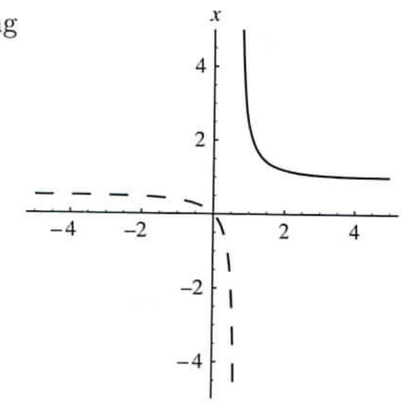
15. From $y = e^{3x}\cos 2x$ we obtain $y' = 3e^{3x}\cos 2x - 2e^{3x}\sin 2x$ and $y'' = 5e^{3x}\cos 2x - 12e^{3x}\sin 2x$, so that $y'' - 6y' + 13y = 0$.

18. Since $\tan x$ is not defined for $x = \pi/2 + n\pi$, n an integer, the domain of $y = 5\tan 5x$ is $\{x \mid 5x \neq \pi/2 + n\pi\}$
or $\{x \mid x \neq \pi/10 + n\pi/5\}$. From $y' = 25\sec^2 5x$ we have

$$y' = 25(1 + \tan^2 5x) = 25 + 25\tan^2 5x = 25 + y^2.$$

An interval of definition for the solution of the differential equation is $(-\pi/10, \pi/10)$. Another interval is $(\pi/10, 3\pi/10)$, and so on.

21. Writing $\ln(2X-1)-\ln(X-1)=t$ and differentiating implicitly we obtain



$$\frac{2}{2X-1}\frac{dX}{dt}-\frac{1}{X-1}\frac{dX}{dt}=1$$

$$\left(\frac{2}{2X-1}-\frac{1}{X-1}\right)\frac{dX}{dt}=1$$

$$\frac{2X-2-2X+1}{(2X-1)(X-1)}\frac{dX}{dt}=1$$

$$\frac{dX}{dt}=-(2X-1)(X-1)=(X-1)(1-2X).$$

Exponentiating both sides of the implicit solution we obtain

$$\frac{2X-1}{X-1}=e^t$$

$$2X-1=Xe^t-e^t$$

$$(e^t-1)=(e^t-2)X$$

$$X=\frac{e^t-1}{e^t-2}.$$

Solving $e^t-2=0$ we get $t=\ln 2$. Thus the solution is defined on $(-\infty,\ln 2)$ or on $(\ln 2,\infty)$. The graph of the solution defined on $(-\infty,\ln 2)$ is dashed, and the graph of the solution defined on $(\ln 2,\infty)$ is solid.

24. Differentiating $y=2x^2-1+c_1e^{-2x^2}$ we obtain $\dfrac{dy}{dx}=4x-4xc_1e^{-2x^2}$, so that

$$\frac{dy}{dx}+4xy=4x-4xc_1e^{-2x^2}+8x^3-4x+4c_1xe^{-x^2}=8x^3$$

In Problems 27–30, we use the Product Rule and the derivative of an integral ((12) of this section): $\dfrac{d}{dx}\displaystyle\int_a^x g(t)\,dt=g(x)$.

27. Differentiating $y=e^{3x}\displaystyle\int_1^x\frac{e^{-3t}}{t}\,dt$ we obtain $\dfrac{dy}{dx}=e^{3x}\displaystyle\int_1^x\frac{e^{-3t}}{t}\,dt+\frac{e^{-3t}}{x}\cdot e^{3x}$ or $\dfrac{dy}{dx}=e^{3x}\displaystyle\int_1^x\frac{e^{-3t}}{t}\,dt+\frac{1}{x}$, so that

$$x\frac{dy}{dx}-3xy=x\left(e^{3x}\int_1^x\frac{e^{-3t}}{t}\,dt+\frac{1}{x}\right)-3x\left(e^{3x}\int_1^x\frac{e^{-3t}}{t}\,dt\right)$$

$$=xe^{3x}\int_1^x\frac{e^{-3t}}{t}\,dt+1-3xe^{3x}\int_1^x\frac{e^{-3t}}{t}\,dt=1$$

30. Differentiating $y = e^{-x^2} + e^{-x^2}\int_0^x e^{t^2}\,dt$ we obtain $\dfrac{dy}{dx} = -2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + e^{x^2} \cdot e^{-x^2}$ or $\dfrac{dy}{dx} = -2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + 1$, so that

$$\frac{dy}{dx} + 2xy = \left(-2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + 1\right) + 2x\left(e^{-x^2} + e^{-x^2}\int_0^x e^{t^2}\,dt\right)$$
$$= -2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + 1 + 2xe^{-x^2} + 2xe^{-x^2}\int_0^x e^{t^2}\,dt = 1$$

33. Force the function $y = e^{mx}$ into the equation $y' + 2y = 0$ to get

$$(e^{mx})' + 2(e^{mx}) = 0$$
$$me^{mx} + 2e^{mx} = 0$$
$$e^{mx}(m+2) = 0$$

Now since $e^{mx} > 0$ for all values of x, we must have $m = -2$ and so $y = e^{-2x}$ is a solution.

36. Force the function $y = e^{mx}$ into the equation $2y'' + 9y' - 5y = 0$ to get

$$2(e^{mx})'' + 9(e^{mx})' - 5(e^{mx}) = 0$$
$$2m^2e^{mx} + 9me^{mx} - 5e^{mx} = 0$$
$$e^{mx}(2m^2 + 9m - 5) = 0$$
$$e^{mx}(m+5)(2m-1) = 0$$

Now since $e^{mx} > 0$ for all values of x , we must have $m = -5$ and $m = 1/2$ therefore $y = e^{-5x}$ and $y = e^{x/2}$ are solutions.

39. Force the function $y = x^m$ into the equation $x^2y'' - 7xy' + 15y = 0$ to get

$$x^2 \cdot (x^m)'' - 7x \cdot (x^m)' + 15(x^m) = 0$$
$$x^2 \cdot m(m-1)x^{m-2} - 7x \cdot mx^{m-1} + 15x^m = 0$$
$$(m^2 - m)x^m - 7mx^m + 15x^m = 0$$
$$x^m[m^2 - 8m + 15] = 0$$
$$x^m[(m-3)(m-5)] = 0$$

The last line implies that $m = 3$ and $m = 5$ therefore $y = x^3$ and $y = x^5$ are solutions.

In Problems 41–44, we substitute $y = c$ into the differential equations and use $y' = 0$ and $y'' = 0$

42. Solving $c^2 + 2c - 3 = (c+3)(c-1) = 0$ we see that $y = -3$ and $y = 1$ are constant solutions.

45. From $y = (x + c_1)^2$ we obtain

$$\left(\frac{dy}{dx}\right)^2 = (2\,(x + c_1))^2 = 4\,(x + c_1)^2.$$

Then

$$4y = 4(x + c_1)^2.$$

Inspection of the differential equation reveals that $y = 0$ is a solution of the differential equation but is not a member of the one-parameter family $y = (x + c_1)^2$.

48. From $y = x - (x - c_1)^2$ we obtain

$$\left(\frac{dy}{dx}\right)^2 = (1 - 2\,(x - c_1))^2 = 1 - 4\,(x - c_1) + 4\,(x - c_1)^2 = 1 - 4x + 4c_1 + 4x^2 - 8xc_1 + 4c_1^2$$

$$-2\frac{dy}{dx} = -2\,(1 - 2\,(x - c_1)) = -2 + 4x - 4c_1$$

$$4y = 4\left(x - (x - c_1)^2\right) = 4\left(x - x^2 + 2xc_1 - c_1^2\right) = 4x - 4x^2 + 8xc_1 - 4c_1^2.$$

Then

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 4y = 4x - 1.$$

Inspection of the differential equation reveals that $y = x$ is a solution of the differential equation but not a member of the one-parameter family $y = x - (x - c_1)^2$.

1.2 Initial-Value Problems

3. Letting $x = 2$ and solving $1/3 = 1/(4 + c)$ we get $c = -1$. The solution is $y = 1/(x^2 - 1)$. This solution is defined on the interval $(1, \infty)$.

6. Letting $x = 1/2$ and solving $-4 = 1/(1/4 + c)$ we get $c = -1/2$. The solution is $y = 1/(x^2 - 1/2) = 2/(2x^2 - 1)$. This solution is defined on the interval $(-1/\sqrt{2}\,, 1/\sqrt{2}\,)$.

In Problems 7–10, we use $x = c_1 \cos t + c_2 \sin t$ and $x' = -c_1 \sin t + c_2 \cos t$ to obtain a system of two equations in the two unknowns c_1 and c_2.

9. From the initial conditions we obtain

$$\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2} \quad -\frac{1}{2}c_2 + \frac{\sqrt{3}}{2} = 0$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is

$$x = (\sqrt{3}/4)\cos t + (1/4)\sin t.$$

In Problems 11–14, we use $y = c_1e^x + c_2e^{-x}$ and $y' = c_1e^x - c_2e^{-x}$ to obtain a system of two equations in the two unknowns c_1 and c_2.

12. From the initial conditions we obtain

$$ec_1 + e^{-1}c_2 = 0$$
$$ec_1 - e^{-1}c_2 = e.$$

Solving, we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}e^2$. The solution of the initial-value problem is

$$y = \frac{1}{2}e^x - \frac{1}{2}e^2e^{-x} = \frac{1}{2}e^x - \frac{1}{2}e^{2-x}.$$

15. Two solutions are $y = 0$ and $y = x^3$.

18. For $f(x,y) = \sqrt{xy}$ we have $\partial f/\partial y = \frac{1}{2}\sqrt{x/y}$. Thus the differential equation will have a unique solution in any region where $x > 0$ and $y > 0$ or where $x < 0$ and $y < 0$.

21. For $f(x,y) = x^2/(4-y^2)$ we have $\partial f/\partial y = 2x^2y/(4-y^2)^2$. Thus the differential equation will have a unique solution in any region where $y < -2$, $-2 < y < 2$, or $y > 2$.

24. For $f(x,y) = (y+x)/(y-x)$ we have $\partial f/\partial y = -2x/(y-x)^2$. Thus the differential equation will have a unique solution in any region where $y < x$ or where $y > x$.

In Problems 25–28, we identify $f(x,y) = \sqrt{y^2-9}$ and $\partial f/\partial y = y/\sqrt{y^2-9}$. We see that f and $\partial f/\partial y$ are both continuous in the regions of the plane determined by $y < -3$ and $y > 3$ with no restrictions on x.

27. Since $(2,-3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(2,-3)$.

30. (a) Since $\frac{d}{dx}\tan(x+c) = \sec^2(x+c) = 1+\tan^2(x+c)$, we see that $y = \tan(x+c)$ satisfies the differential equation.

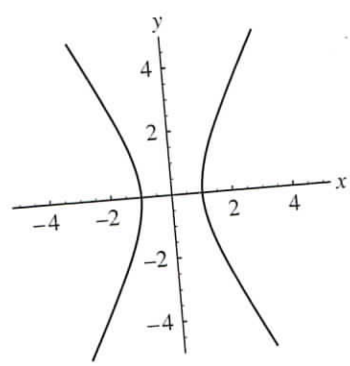
(b) Solving $y(0) = \tan c = 0$ we obtain $c = 0$ and $y = \tan x$. Since $\tan x$ is discontinuous at $x = \pm\pi/2$, the solution is not defined on $(-2,2)$ because it contains $\pm\pi/2$.

(c) The largest interval on which the solution can exist is $(-\pi/2, \pi/2)$.

33. (a) Differentiating $3x^2 - y^2 = c$ we get $6x - 2yy' = 0$ or $yy' = 3x$.

(b) Solving $3x^2 - y^2 = 3$ for y we get

$$y = \phi_1(x) = \sqrt{3(x^2-1)}, \qquad 1 < x < \infty,$$
$$y = \phi_2(x) = -\sqrt{3(x^2-1)}, \qquad 1 < x < \infty,$$
$$y = \phi_3(x) = \sqrt{3(x^2-1)}, \qquad -\infty < x < -1,$$
$$y = \phi_4(x) = -\sqrt{3(x^2-1)}, \qquad -\infty < x < -1.$$



(c) Only $y = \phi_3(x)$ satisfies $y(-2) = 3$.

In Problems 35–38, we consider the points on the graphs with x-coordinates $x_0 = -1$, $x_0 = 0$, and $x_0 = 1$. The slopes of the tangent lines at these points are compared with the slopes given by $y'(x_0)$ in (a) through (f).

36. The graph satisfies the conditions in (e).

39. Using the function $y = c_1 \cos 3x + c_2 \sin 3x$ and the first boundary condition we get

$$y(0) = c_1 \cos 0 + c_2 \sin 0 = 0$$

Therefore $c_1 = 0$. Similarly, for the second boundary condition we get

$$y(\pi/6) = c_2 \sin 3\,(\pi/6) = -1$$

Therefore $c_2 = -1$. The solution to the boundary value problem is $y(x) = -\sin 3x$.

42. The derivative of the function $y = c_1 \cos 3x + c_2 \sin 3x$ is $y' = -3c_1 \sin 3x + 3c_2 \cos 3x$ and using the two boundary conditions we get

$$y(0) = c_1 + 0 = 1$$

Therefore $c_1 = 1$. In addition

$$y'(\pi) = 0 - 3c_2 = 5$$

Therefore $c_2 = -5/3$. The solution to this boundary value problem is $y(x) = \cos 3x - \dfrac{5}{3}\sin 3x$.

1.3 Differential Equations as Mathematical Models

3. Let b be the rate of births and d the rate of deaths. Then $b = k_1 P$ and $d = k_2 P^2$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1 P - k_2 P^2$.

6. By inspecting the graph in the text we take T_m to be $T_m(t) = 80 - 30\cos(\pi t/12)$. Then the temperature of the body at time t is determined by the differential equation

$$\frac{dT}{dt} = k\left[T - \left(80 - 30\cos\left(\frac{\pi}{12}t\right)\right)\right], \quad t > 0.$$

9. The rate at which salt is leaving the tank is

$$R_{out}\ (3\ \text{gal/min}) \cdot \left(\frac{A}{300}\ \text{lb/gal}\right) = \frac{A}{100}\ \text{lb/min}.$$

Thus $dA/dt = A/100$. The initial amount is $A(0) = 50$.

12. The rate at which salt is entering the tank is

$$R_{in} = (c_{in} \text{ lb/gal}) \cdot (r_{in} \text{ gal/min}) = c_{in} r_{in} \text{ lb/min}.$$

Now let $A(t)$ denote the number of pounds of salt and $N(t)$ the number of gallons of brine in the tank at time t. The concentration of salt in the tank as well as in the outflow is $c(t) = x(t)/N(t)$. But the number of gallons of brine in the tank remains steady, is increased, or is decreased depending on whether $r_{in} = r_{out}$, $r_{in} > r_{out}$, or $r_{in} < r_{out}$. In any case, the number of gallons of brine in the tank at time t is $N(t) = N_0 + (r_{in} - r_{out})t$. The output rate of salt is then

$$R_{out} = \left(\frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/gal}\right) \cdot (r_{out} \text{ gal/min}) = r_{out} \frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/min}.$$

The differential equation for the amount of salt, $dA/dt = R_{in} - R_{out}$, is

$$\frac{dA}{dt} = c_{in} r_{in} - r_{out} \frac{A}{N_0 + (r_{in} - r_{out})t} \quad \text{or} \quad \frac{dA}{dt} + \frac{r_{out}}{N_0 + (r_{in} - r_{out})t} A = c_{in} r_{in}.$$

15. Since $i = dq/dt$ and $L\, d^2q/dt^2 + R\, dq/dt = E(t)$, we obtain $L\, di/dt + Ri = E(t)$.

18. Since the barrel in Figure 1.3.17(b) in the text is submerged an additional y feet below its equilibrium position, the number of cubic feet in the additional submerged portion is the volume of the circular cylinder: $\pi \times (\text{radius})^2 \times \text{height}$ or $\pi(s/2)^2 y$. Then we have from Archimedes' principle

$$\begin{aligned} \text{Upward force of water on barrel} &= \text{Weight of water displaced} \\ &= (62.4) \times (\text{Volume of water displaced}) \\ &= (62.4)\pi(s/2)^2 y = 15.6\pi s^2 y. \end{aligned}$$

It then follows from Newton's second law that

$$\frac{w}{g}\frac{d^2y}{dt^2} = -15.6\pi s^2 y \quad \text{or} \quad \frac{d^2y}{dt^2} + \frac{15.6\pi s^2 g}{w} y = 0,$$

where $g = 32$ and w is the weight of the barrel in pounds.

21. As the rocket climbs (in the positive direction), it spends its amount of fuel and therefore the mass of the fuel changes with time. The air resistance acts in the opposite direction of the motion and the upward thrust R works in the same direction. Using Newton's second law we get

$$\frac{d}{dt}(mv) = -mg - kv + R$$

Now because the mass is variable, we must use the product rule to expand the left side of the equation. Doing so gives us the following:

$$\frac{d}{dt}(mv) = -mg - kv + R$$

$$v \times \frac{dm}{dt} + m \times \frac{dv}{dt} = -mg - kv + R$$

The last line is the differential equation we wanted to find.

24. The gravitational force on m is $F = -kM_r m/r^2$. Since $M_r = 4\pi\delta r^3/3$ and $M = 4\pi\delta R^3/3$, we have $M_r = r^3 M/R^3$ and

$$F = -k\frac{M_r m}{r^2} = -k\frac{r^3 Mm/R^3}{r^2} = -k\frac{mM}{R^3}r.$$

Now from $F = ma = d^2r/dt^2$ we have

$$m\frac{d^2r}{dt^2} = -k\frac{mM}{R^3}r \quad \text{or} \quad \frac{d^2r}{dt^2} = -\frac{kM}{R^3}r.$$

27. The differential equation is $x'(t) = r - kx(t)$ where $k > 0$.

Chapter 1 in Review

3. $\dfrac{d}{dx}(c_1\cos kx + c_2\sin kx) = -kc_1\sin kx + kc_2\cos kx;$

$$\frac{d^2}{dx^2}(c_1\cos kx + c_2\sin kx) = -k^2c_1\cos kx - k^2c_2\sin kx = -k^2\overbrace{(c_1\cos kx + c_2\sin kx)}^{y};$$

$$\frac{d^2y}{dx^2} = -k^2y \quad \text{or} \quad \frac{d^2y}{dx^2} + k^2y = 0$$

6. $y' = -c_1e^x\sin x + c_1e^x\cos x + c_2e^x\cos x + c_2e^x\sin x;$

$y'' = -c_1e^x\cos x - c_1e^x\sin x - c_1e^x\sin x + c_1e^x\cos x - c_2e^x\sin x + c_2e^x\cos x + c_2e^x\cos x + c_2e^x\sin x$

$= -2c_1e^x\sin x + 2c_2e^x\cos x;$

$y'' - 2y' = -2c_1e^x\cos x - 2c_2e^x\sin x = -2y; \qquad y'' - 2y' + 2y = 0$

9. b **12.** a, b, d

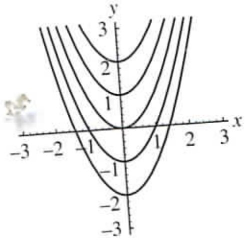
15. The slope of the tangent line at (x, y) is y', so the differential equation is $y' = x^2 + y^2$.

18. **(a)** Differentiating $y^2 - 2y = x^2 - x + c$ we obtain $2yy' - 2y' = 2x - 1$ or $(2y - 2)y' = 2x - 1$.

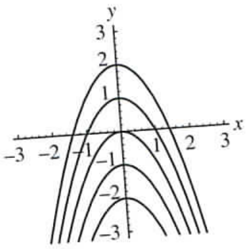
(b) Setting $x = 0$ and $y = 1$ in the solution we have $1 - 2 = 0 - 0 + c$ or $c = -1$. Thus, a solution of the initial-value problem is $y^2 - 2y = x^2 - x - 1$.

(c) Solving the equation $y^2 - 2y - (x^2 - x - 1) = 0$ by the quadratic formula we get $y = (2 \pm \sqrt{4 + 4(x^2 - x - 1)}\,)/2 = 1 \pm \sqrt{x^2 - x} = 1 \pm \sqrt{x(x-1)}$. Since $x(x-1) \geq 0$ for $x \leq 0$ or $x \geq 1$, we see that neither $y = 1 + \sqrt{x(x-1)}$ nor $y = 1 - \sqrt{x(x-1)}$ is differentiable at $x = 0$. Thus, both functions are solutions of the differential equation, but neither is a solution of the initial-value problem.

21. (a)



$y = x^2 + c_1$



$y = -x^2 + c_2$

(b) When $y = x^2 + c_1$, $y' = 2x$ and $(y')^2 = 4x^2$. When $y = -x^2 + c_2$, $y' = -2x$ and $(y')^2 = 4x^2$.

(c) Pasting together x^2, $x \geq 0$, and $-x^2$, $x \leq 0$, we get $y = \begin{cases} -x^2, & x \leq 0 \\ x^2, & x > 0\,. \end{cases}$

24. Differentiating $y = x \sin x + (\cos x) \ln(\cos x)$ we get

$$\begin{aligned} y' &= x \cos x + \sin x + \cos x \left(\frac{-\sin x}{\cos x} \right) - (\sin x) \ln(\cos x) \\ &= x \cos x + \sin x - \sin x - (\sin x) \ln(\cos x) \\ &= x \cos x - (\sin x) \ln(\cos x) \end{aligned}$$

and

$$\begin{aligned} y'' &= -x \sin x + \cos x - \sin x \left(\frac{-\sin x}{\cos x} \right) - (\cos x) \ln(\cos x) \\ &= -x \sin x + \cos x + \frac{\sin^2 x}{\cos x} - (\cos x) \ln(\cos x) \\ &= -x \sin x + \cos x + \frac{1 - \cos^2 x}{\cos x} - (\cos x) \ln(\cos x) \\ &= -x \sin x + \cos x + \sec x - \cos x - (\cos x) \ln(\cos x) \\ &= -x \sin x + \sec x - (\cos x) \ln(\cos x). \end{aligned}$$

Thus

$$y'' + y = -x \sin x + \sec x - (\cos x) \ln(\cos x) + x \sin x + (\cos x) \ln(\cos x) = \sec x.$$

To obtain an interval of definition we note that the domain of $\ln x$ is $(0, \infty)$, so we must have $\cos x > 0$. Thus, an interval of definition is $(-\pi/2, \pi/2)$.

In Problems 27–30 we use (12) of Section 1.1 and the Product Rule.

27.

$$\begin{aligned} y &= e^{\cos x} \int_0^x t e^{-\cos t}\, dt \\ \frac{dy}{dx} &= e^{\cos x} \left(x e^{-\cos x} \right) - \sin x\, e^{\cos x} \int_0^x t e^{-\cos t}\, dt \end{aligned}$$

$$\frac{dy}{dx} + (\sin x)\,y = e^{\cos x} x e^{-\cos x} - \sin x e^{\cos x} \int_0^x t e^{-\cos t}\,dt + \sin x\left(e^{\cos x}\int_0^x t e^{-\cos t}\,dt\right)$$
$$= x - \sin x e^{\cos x}\int_0^x t e^{-\cos t}\,dt + \sin x e^{\cos x}\int_0^x t e^{-\cos t}\,dt = x$$

30.

$$y = \sin x \int_0^x e^{t^2}\cos t\,dt - \cos x \int_0^x e^{t^2}\sin t\,dt$$
$$y' = \sin x\left(e^{x^2}\cos x\right) + \cos x\int_0^x e^{t^2}\cos t\,dt - \cos x\left(e^{x^2}\sin x\right) + \sin x\int_0^x e^{t^2}\sin t\,dt$$
$$= \cos x\int_0^x e^{t^2}\cos t\,dt + \sin x\int_0^x e^{t^2}\sin t\,dt$$
$$y'' = \cos x\left(e^{x^2}\cos x\right) - \sin x\int_0^x e^{t^2}\cos t\,dt + \sin x\left(e^{x^2}\sin x\right) + \cos x\int_0^x e^{t^2}\sin t\,dt$$
$$= e^{x^2}\left(\cos^2 x + \sin^2 x\right) - \left(\overbrace{\sin x\int_0^x e^{t^2}\cos t\,dt - \cos x\int_0^x e^{t^2}\sin t\,dt}^{y}\right)$$
$$= e^{x^2} - y$$

$$y'' + y = e^{x^2} - y + y = e^{x^2}$$

33. Using implicit differentiation we get

$$y^3 + 3y = 2 - 3x$$
$$3y^2y' + 3y' = -3$$
$$y^2y' + y' = -1$$
$$(y^2+1)y' = -1$$
$$y' = \frac{-1}{y^2+1}$$

Differentiating the last line and remembering to use the quotient rule on the right side leads to

$$y'' = \frac{2yy'}{(y^2+1)^2}$$

Now since $y' = -1\big/\left(y^2+1\right)$ we can write the last equation as

$$y'' = \frac{2y}{(y^2+1)^2}y' = \frac{2y}{(y^2+1)^2}\frac{-1}{(y^2+1)} = 2y\left(\frac{-1}{y^2+1}\right)^3 = 2y(y')^3$$

which is what we wanted to show.

36. Substituting $y = c_1 + c_2 x$ and $y' = c_2$ into the left-hand side of the differential equation gives

$$y' + 2y = c_2 + 2(c_1 + c_2 x) = c_2 + 2c_1 + 2c_2 x.$$

Setting the result equal to the right-hand side of the differential equation yields

$$c_2 + 2c_1 + 2c_2 x = 3x$$
$$(c_2 + 2c_1) + 2c_2 x = 0 + 3x.$$

Therefore, $2c_2 = 3$ or $c_2 = \dfrac{3}{2}$, and $\dfrac{3}{2} + 2c_1 = 0$ or $c_1 = -\dfrac{3}{2} \cdot \dfrac{1}{2} = -\frac{3}{4}$. Thus $y = -\dfrac{3}{4} + \dfrac{3}{2}x$.

In Problem 39–42, $y = c_1 e^{-3x} + c_2 e^{x} + 4x$ is given as a two-parameter family of solutions of the second-order differential equation $y'' + 2y' - 3y = -12x + 8$.

39. If $y(0) = 0$ and $y'(0) = 0$, then

$$c_1 + c_2 = 0$$
$$-3c_1 + c_2 = -4$$

subtracting the second equation from the first gives us $4c_1 = 4$ or $c_1 = 1$, and thus $c_2 = -1$. Therefore $y = e^{-3x} - e^{x} + 4x$.

42. If $y(-1) = 1$ and $y'(-1) = 1$, then

$$c_1 e^{3} + c_2 e^{-1} = 5$$
$$-3c_1 e^{3} + c_2 e^{-1} = -3$$

subtracting the second equation from the first gives us $4c_1 = 8$ or $c_1 = 2e^{-3}$, and thus $c_2 = 3e$. Therefore $y = 2e^{-3x-3} + 3e^{x+1} + 4x$.

45. From the graph we see that estimates for y_0 and y_1 are $y_0 = -3$ and $y_1 = 0$.